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Average complexity of the Best Response Algorithm in Potential Games

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1. Introduction

The computation of Nash Equilibria (NE) in games has been investigated of many papers. The most general result is in [2] and says that the problem of computing NE is PPAD complete.

Potential games have been introduced in [7] and have proven very useful, especially in the context of routing games, first mentioned in [1] and exhaustively studied ever since, in the transportation as well as computer science litterature, see for example [6]. For potential games, efficient polynomial time algorithms exist in symmetric cases (see [4]). However, the same paper shows that the computation of NE for general potential games is PLS complete. The Best Response Algorithm (BRA) is probably the most popular algorithm that converges to a pure Nash equilibrium (NE) in potential games [5]. However, its complexity has attracted surprisingly little attention.

In this paper, we analyze the performance of BRA over a potential game with N players, each with A possible actions. We show that on average, the Best Response Algorithm takes $\log(N)+e^{\gamma}$ (γ is the Euler constant) effectives steps and makes $e^{\gamma}AN$ comparisons before finding a NE.

These numbers say that BRA is very efficient on average to compute NE, even if this is a PLE complete problem. Our analysis is based on two ingredients, one is the construction of an approximation of the behavior of BRA, where each state is examined at most once and the second is the use of a continuous space discrete time Markov chain to analyze the average complexity.

2. Best Response Algorithm and Potential games

We consider a game with a finite number N of players and a finite strategy space for each player, each of size A, and the corresponding utility functions. The game $\mathfrak{G} \stackrel{\text{def}}{=} \mathfrak{G}(\mathcal{N}, \mathcal{A}, u)$ will be a tuple consisting of

- a finite set of *players* $\mathcal{N} = \{1, \dots, N\}$;
- a finite set $\mathcal{A}_k$ of *actions* (or *pure strategies*) for each player $k \in \mathcal{N}$; The set of *action profiles* or *states* of the game is $\mathcal{A} \stackrel{\text{def}}{=} \prod_k \mathcal{A}_k$;
- the players' *payoff functions* $u_k : \mathcal{A} \to \mathbb{R}$.

We define the classical *best response correspondence* $\mathbf{br}_k(x)$ as the set of all actions that maximizes the payoff for player k under profile x :

$$\mathbf{br}_k(x) \stackrel{\text{def}}{=} \left\{ \underset{\alpha \in \mathcal{A}_k}{\operatorname{argmax}}\, u_k(\alpha; x_{-k}) \right\}. \tag{1}$$

A *Nash equilibrium* (NE) is a fixed point of the correspondence, i.e. a profile x^* such that $x^*_k \in \mathbf{br}_k(x^*)$ for every player k.

Definition 1 (Potential games) *A game is a* potential game *[5] if it admits a function (called the potential)* $\Phi : \mathcal{A} \to \mathbb{R}$ *such that for any player* k *and any* unilateral *deviation of* k *from action* α *to* α', $u_k(\alpha, x_{-k}) - u_k(\alpha', x_{-k}) = \Phi(\alpha, x_{-k}) - \Phi(\alpha', x_{-k})$.

We consider a version of *Best Response Algorithm* (BRA) where the next player is selected according to a round robin pattern. Other patterns can also be considered using the same approach and can be shown to have a similar behavior.

Algorithm 1: Best Response Algorithm (BRA)

Input :
Game utilities $(u_i(\cdot))$,
Initial state $(x(0))$,
Infinite seq. of players $R = (1, 2, \dots, N, 1, \dots)$.

foreach *player* $k \in K$ **do**
  $\mathsf{stop}_k := \mathsf{false}$
repeat
  Pick next player $k := R_{t+1}$
  Select new action $\alpha_k := \mathbf{br}_k(x(t))$
  $\mathsf{stop}_k := \mathbf{1}_{\{\alpha_k = x_k(t)\}}$;
  $x_k(t+1) := \alpha_k$;
until $\mathsf{stop}_1 \wedge \mathsf{stop}_2 \wedge \cdots \wedge \mathsf{stop}_N$;

A famous result first proved in [5] states that for any potential game $\mathfrak{G}$, Algorithm 1 converges in finite time to a Nash Equilibrium of $\mathfrak{G}$.

3. Complexity

Let us consider three complexity measures (related to each other) : T_{BRA} is the number of iterations (or the number of times that the function **br** was called) before BRA reaches a Nash equilibrium. The total number of comparisons is denoted C_{BRA}. One should expect that $C_{BRA} \approx (A-1)T_{BRA}$. Finally, the number of different states visited by BRA is denoted M_{BRA}. Of course, $M_{BRA} \leqslant T_{BRA}$. The proofs of Theorems 1 and 2 are not provided due to lack of space. They are available in a research report [3].

Theorem 1 *In the worst case, under round robin revisions,* $T_{BRA} = NA^{N-1}$.

3.1. Randomization

In the following we will analyze the average complexity of BRA.

We randomize over all the potential games over which BRA is used. Since the behavior of BRA only depends on the potential function, we randomize directly over the potential Φ. The natural randomization is to consider all possible total orderings of the set $\{\Phi(x), x \in \mathcal{A}\}$ (there are $(A^N)!$ of them) and pick one uniformly. This is equivalent to pick iid potentials in all states, uniformly distributed in $[0,1]$.

3.2. Markovian Analysis

We will be analyzing the intersection-free approximation of the behavior of BRA (where no state is visited twice) whose behavior is asymptotically the same as BRA.

Let y be the potential of the current state x : ($y \stackrel{\text{def}}{=} \Phi(x)$). If $k-1$ players have already played best response without changing the state, then the evolution at the next step of BRA is as follows. The k-th player computes its best response. This player has $a \stackrel{\text{def}}{=} A-1$ new actions whose potential must be compared with the current potential (y). With probability y^a none of the new actions beat the current choice. The state remains at y and it is the turn of the $k+1$-st player to try its best response. With probability $1-y^a$, one of the new actions is the best response. The current state moves to a new state with a larger potential and the number of players for which the new state is a best response is set back to 1.

This says that the couple (Y_t, K_t) is a Markov chain, where Y_t is the potential at step t, in $[0,1]$ and K_t is the current number of players whose best response did not change the current state (in $\{1, 2, \ldots, N\}$). Its transitions are :

$$\mathbb{P}\left((Y,K)_{t+1} = (y, k+1) \middle| (Y,K)_t = (y,k)\right) = y^a,$$

and, if $z > y$,

$$\mathbb{P}\left((Y,K)_{t+1} \in ([z,1],1) \middle| (Y,K)_t = (y,k)\right) = 1 - z^a.$$

Let $C(y,k)$ be the average number of comparisons required to reach a NE, starting in a state with potential y where k players have played without changing their action. The forward equation for $C(y,k)$ is :

$$\begin{aligned} C(y,k) =& y^a(C(y,k+1) + a) \\ &+ \int_y^1 a u^{a-1}(C(u,1) + a) du, \end{aligned}$$

with the boundary conditions $C(1,1) = a(N-1)$ and $C(y,N) = 0$.

Solving these equations leads to the following proposition (quantities M_{BRA} and T_{BRA} are analyzed similarly).

Theorem 2 *The average number of moves in BRA verifies* $\mathbb{E}M_{BRA} \leqslant \log(N) + e^\gamma + O(1/N)$.
The average number of comparisons verifies
$\mathbb{E}C_{BRA} \leqslant e^\gamma (A-1)N + o(A)$
and the average number of steps verifies
$\mathbb{E}T_{BRA} \leqslant e^\gamma N + o(1)$.

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