A Mean-Field Game with Explicit Interactions for Epidemic Models

Josu Doncel, Nicolas Gast, Bruno Gaujal *

INRIA Grenoble Rhônes-Alpes
655 Avenue de l'Europe
38330 Montbonnot-Saint-Martin
{josu.doncel, nicolas.gast,bruno.gaujal}@inria.fr

1. Introduction

Game theory studies the rational behavior of decision-makers (called players in the following). A crucial notion is the concept of Nash equilibrium. A Nash equilibrium is an allocation of strategies such that no player can benefit from unilateral deviation. Although any finite game has at least one Nash equilibrium, it is shown in [1] that computing a Nash equilibrium is a PPAD-complete [2] problem. This suggests that the computation of a Nash equilibrium is not tractable when the number of players or of strategies is large. As an alternative, the notion of mean-field games has been introduced by Lasry and Lions in [6], which is a game where an individual object is infinitesimal and does not affect the global system behavior.

Here, we study mean field games with two specific features : Each player has a finite state space (instead of a continuous one in [6]), and the dynamics of one player depends explicitly on the behavior of the others (unlike in most works in mean field games [6, 3]). The general theory is developed in [2]. In this document, we illustrate the theory with a vaccination problem.

2. Model Description

2.1. Epidemic Model

We consider a population of N homogeneous objects that evolve in continuous time from 0 to T. The objects can be susceptible, infected, recovered or vaccinated. We denote by $S(t)$, $I(t)$, $R(t)$ and $V(t)$ the proportion of the population that is, respectively, susceptible, infected, recovered and vaccinated at time t.

The dynamics of one object is a Markov process that can be described as follows. An object encounters other objects with rate β. If the initial object was susceptible and the encounter was infected, the first object becomes infected. An infected object recovers at rate γ. We also consider that there is a vaccination policy $\mathbf{b}$ that is applied to each object of the susceptible population. The vaccination rate $\mathbf{b}$ is a function from 0 to T that takes values in the interval $[0, b_{max}]$. Once an object is vaccinated or recovered, it does not change its state. The dynamics of an object is described in Figure 1.

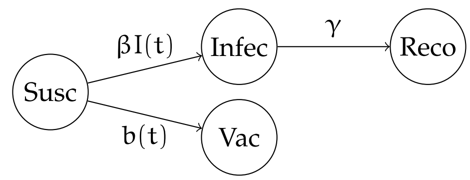

FIGURE 1 – The dynamics of an object in the epidemic model.

We are interested in the analysis of this epidemic model for a large number of objects. When $N \to \infty$, the dynamics of the population converges [4] to the following system of differential equations :

$$\begin{cases} \dot{S}(t) = -\beta \cdot S(t) \cdot I(t) - b(t) \cdot S(t) \\ \dot{I}(t) = \beta \cdot S(t) \cdot I(t) - \gamma \cdot I(t) \\ \dot{R}(t) = \gamma \cdot I(t) \\ \dot{V}(t) = b(t) \cdot S(t) \end{cases} \tag{1}$$

In [5] the authors develop an approximation of this epidemic model and characterize the solution of the derived mean-field game. In the rest of the paper, we show that the mean-field game corresponding to this model is tractable and can be analyzed rigorously.

2.2. Mean-Field Game

We focus on a particular object, that we call Player 0. Let $X(t) \in \{Sus, Infec, Reco, Vac\}$ be the state of Player 0 at time t. We note that the evolution of $X(t)$ depends on the infected population. We assume that the rest of the population applies a fixed vaccination policy $\mathbf{b}$. Player 0 chooses its vaccination policy $\mathbf{b}_0$, so as to minimize its expected individual cost, which is

$$C_{ind}(\mathbf{b}_0, \mathbf{b}) = \int_0^T (c_V b_0(t) \mathbb{P}(X(t) = Susc) + c_I \mathbb{P}(X(t) = Infec)) dt,$$

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2. PPAD stands for "polynomial parity arguments on directed graphs". It is a complexity class that is a subclass of NP and is believed to be strictly greater than P.

where c_V is the vaccination cost and c_I is the unit time cost of being infected.

We call the *best response to* $\mathbf{b}$ and denote by $\mathrm{BR}(\mathbf{b})$ the set of vaccination policies that minimize the cost of Player 0 for a given vaccination policy of the population $\mathbf{b}$:

Definition 1 (Best-Reponse)

$$\mathrm{BR}(\mathbf{b}) = \arg\min_{\mathbf{b_0}} C_{ind}(\mathbf{b}_0, \mathbf{b}). \tag{2}$$

We now define the notion of a mean-field equilibrium for this game. It is a vaccination strategy $\mathbf{b}^{\mathrm{MFE}}$ such that when the population chooses the vaccination policy $\mathbf{b}^{\mathrm{MFE}}$, a selfish Player 0 would also choose the same vaccination policy $\mathbf{b}^{\mathrm{MFE}}$:

Definition 2 (Symmetric Mean-Field Equilibrium) *The vaccination policy $\mathbf{b}^{\mathrm{MFE}}$ is a symmetric mean-field equilibrium if and only if*

$$\mathbf{b}^{\mathrm{MFE}} \in \mathrm{BR}(\mathbf{b}^{\mathrm{MFE}}).$$

The rationale behind this definition is when one considers that the population is made of players that each take self-interested decisions. As the population is homogeneous, each object best-response is the same as Player 0. In other words, for a given population vaccination policy $\mathbf{b}$, all the objects of the populations choose the strategy $\mathrm{BR}(\mathbf{b})$. A mean-field equilibrium is a situation where no object has incentive to deviate unilaterally from its strategy.

2.3. Centralized Control Problem

The mean-field game scenario corresponds to a case where the decisions are selfish and decentralized. The corresponding centralized control problem can also be defined naturally. For a given vaccination population $\mathbf{b}$,the average system cost is

$$C_{sys}(\mathbf{b}) = \int_0^T (c_V\, b(t)\, S(t) + c_I\, I(t))\ dt.$$

This cost represents the cost in the system when the population vaccination policy is $\mathbf{b}$. A global optimum is a vaccination policy that minimizes the system cost

Definition 3 (Global Optimum)

$$\mathbf{b}^{\mathrm{OPT}} \in \arg\min_{\mathbf{b}} C_{sys}(\mathbf{b}). \tag{3}$$

3. Main Results

The mean-field game we analyze here is a particular case of the model of [2]. In [2], we introduce the mean-field games with explicit interactions, which is a discrete state space model where the transition rates between states depend not only on the actions taken, but also on the empirical measure of the system. This *explicit interactions* between objects makes our model distinct from most work on mean-field games.

To characterize a symmetric mean-field equilibrium of the epidemic model, we model the best-response of the generic object as a Continuous Time Markov Decision Process and we show that, for any population vaccination policy $\mathbf{b}$, the best-response strategy of the generic object is of threshold type. This result yields the following proposition.

Proposition 1 *There exists a symmetric mean-field equilibrium that is pure and of threshold type.*

We also analyze the global optimum of the epidemic model.

Proposition 2 *There exists a global optimum of threshold type.*

Unfortunately, in all but degenerated cases, the thresholds do not coincide, so that the price of anarchy of this model is never equal to 1. Numerical simulations show that the price of anarchy is small in general. A pricing mechanism can be used to force the equilibrium to coincide with the global optimum. Our numerical experiments show that to encourage selfish individuals to vaccinate optimally, vaccination should be subsidized.

Bibliographie

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